

# Chapter 6: The Fermi Liquid

L.D. Landau

January 13, 2008

## Contents

<b>1</b>	<b>introduction: The Electronic Fermi Liquid</b>	<b>3</b>
<b>2</b>	<b>The Non-Interacting Fermi Gas</b>	<b>5</b>
2.1	Infinite-Square-Well Potential . . . . .	5
2.2	The Fermi Gas . . . . .	10
2.2.1	$T = 0$ , The Pauli Principle . . . . .	10
2.2.2	$T \neq 0$ , Fermi Statistics . . . . .	13
<b>3</b>	<b>The Weakly Correlated Electronic Liquid</b>	<b>23</b>
3.1	Thomas-Fermi Screening . . . . .	23
3.2	Fermi liquids . . . . .	28
3.3	Quasi-particles . . . . .	29
3.3.1	Particles and Holes . . . . .	29
3.3.2	Quasiparticles and Quasiholes at $T = 0$ . . . . .	33
3.4	Energy of Quasiparticles. . . . .	39
<b>4</b>	<b>Interactions between Particles: Landau Fermi Liquid</b>	<b>42</b>
4.1	The free energy, and interparticle interactions . . . . .	42
4.2	Local Energy of a Quasiparticle . . . . .	46

4.2.1	Equilibrium Distribution of Quasiparticles at Finite $T$ . . . . .	48
4.2.2	Local Equilibrium Distribution . . . . .	50
4.3	Effective Mass $m^*$ of Quasiparticles . . . . .	52

## 1 introduction: The Electronic Fermi Liquid

As we have seen, the electronic and lattice degrees of freedom decouple, to a good approximation, in solids. This is due to the different time scales involved in these systems.

$$\tau_{ion} \sim 1/\omega_D \gg \tau_{electron} \sim \frac{\hbar}{E_F} \quad (1)$$

where  $E_F$  is the electronic Fermi energy. The electrons may be thought of as instantly reacting to the (slow) motion of the lattice, while remaining essentially in the electronic ground state. Thus, to a good approximation the electronic and lattice degrees of freedom separate, and the small electron-lattice (phonon) interaction (responsible for resistivity, superconductivity etc) may be treated as a perturbation (with  $\omega_D/E_F$  as an expansion parameter); that is if we are capable of solving the problem of the remaining purely electronic system.

At first glance the remaining electronic problem would also appear to be hopeless since the (non-perturbative) electron-electron interactions are as large as the combined electronic kinetic energy and the potential energy due to interactions with

the static ions (the latter energy, or rather the corresponding part of the Hamiltonian, composes the solvable portion of the problem). However, the Pauli principle keeps low-lying orbitals from being multiply occupied, so is often justified to ignore the electron-electron interactions, or treat them as a renormalization of the non-interacting problem (effective mass) etc. This will be the initial assumption of this chapter, in which we will cover

- the non-interacting Fermi liquid, and
- the renormalized Landau Fermi liquid (Pines Nozieres).

These relatively simple theories resolve some of the most important puzzles involving metals at the turn of the century. Perhaps the most intriguing of these is the metallic specific heat. Except in certain “heavy fermion” metals, the electronic contribution to the specific heat is always orders of magnitude smaller than the phonon contribution. However, from the classical theorem of equipartition, if each lattice site contributes just one electron to the conduction band, one would expect the contributions

from these sources to be similar ( $C_{electron} \approx C_{phonon} \approx 3Nrk_B$ ). This puzzle is resolved, at the simplest level: that of the non-interacting Fermi gas.

## 2 The Non-Interacting Fermi Gas

### 2.1 Infinite-Square-Well Potential

We will proceed to treat the electronic degrees of freedom, ignoring the electron-electron interaction, and even the electron-lattice interaction. In general, the electronic degrees of freedom are split into electrons which are bound to their atomic cores with wavefunctions which are essentially atomic, unaffected by the lattice, and those valence (or near valence) electrons which react and adapt to their environment. For the most part, we are only interested in the valence electrons. Their environment described by the potential due to the ions and the core electrons—the core potential. Thus, ignoring the electron-electron interactions, the electronic Hamiltonian is

$$H = \frac{P^2}{2m} + V(\mathbf{r}). \quad (2)$$

As shown in Fig. 1, the core potential  $V(\mathbf{r})$ , like the lattice, is periodic

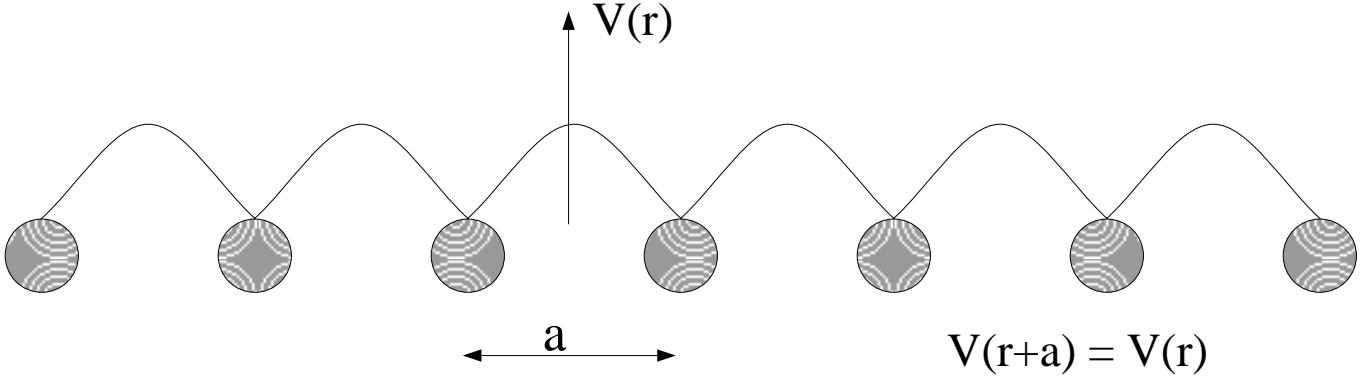


Figure 1: *Schematic core potential (solid line) for a one-dimensional lattice with lattice constant  $a$ .*

For the moment, ignore the core potential, then the electronic wave functions are plane waves  $\psi \sim e^{i\mathbf{k}\cdot\mathbf{r}}$ . Now consider the core potential as a perturbation. The electrons will be strongly effected by the periodicity of the potential when  $\lambda = 2\pi/k \sim a$ <sup>1</sup>. However, when  $\mathbf{k}$  is small so that  $\lambda \gg a$  (or when  $\mathbf{k}$  is large, so  $\lambda \ll a$ ) the structure of the potential may be neglected, or we can assume  $V(\mathbf{r}) = V_0$  anywhere within the material. The

---

<sup>1</sup>Interestingly, when  $\lambda \sim a$ , the Bragg condition  $2d \sin \theta \approx a \approx \lambda$  may easily be satisfied, so the electrons, which may be thought of as DeBroglie waves, scatter off of the lattice. Consequently states for which  $\lambda = 2\pi/k \sim a$  are often forbidden. This is the source of gaps in the band structure, to be discussed in the next chapter.

potential still acts to confine the electrons (and so maintain charge neutrality), so  $V(\mathbf{r}) = \infty$  anywhere outside the material.

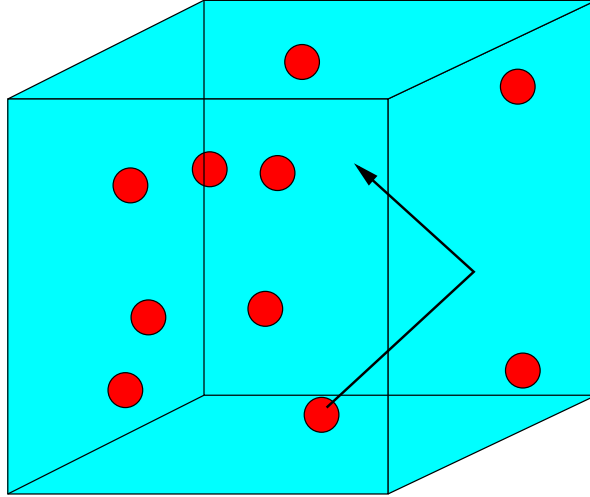


Figure 2: *Infinite square-well potential.*  $V(\mathbf{r}) = V_0$  within the well, and  $V(\mathbf{r}) = \infty$  outside to confine the electrons and maintain charge neutrality.

Thus we will approximate the potential of a cubic solid with linear dimension  $L$  as an infinite square-well potential.

$$V(\mathbf{r}) = \begin{cases} V_0 & 0 < r_i < L \\ \infty & \text{otherwise} \end{cases} \quad (3)$$

The electronic wavefunctions in this potential satisfy

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}) = (E' - V_0) \psi(\mathbf{r}) = E \psi(\mathbf{r}) \quad (4)$$

The normalized plane wave solution to this model is

$$\psi(\mathbf{r}) = \prod_{i=1}^3 \left(\frac{2}{L}\right)^{1/2} \sin k_i x_i \quad \text{where } i = x, y, \text{ or } z \quad (5)$$

and  $k_i L = n_i \pi$  in order to satisfy the boundary condition that  $\psi = 0$  on the surface of the cube. Furthermore, solutions with  $n_i < 0$  are not independent of solutions with  $n_i > 0$  and may be excluded. Solutions with  $n_i = 0$  cannot be normalized and are excluded (they correspond to no electron in the state). The

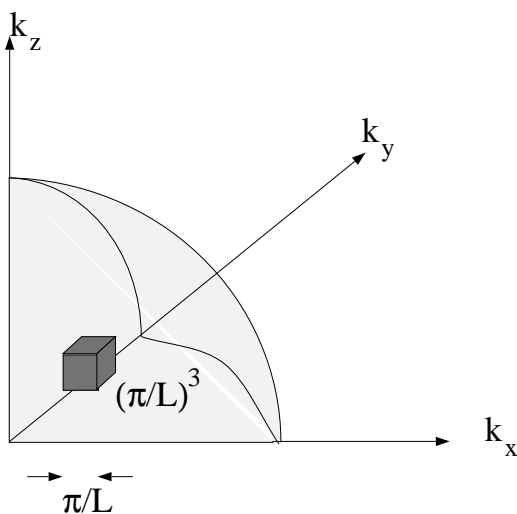


Figure 3: Allowed  $k$ -states for an electron confined by a infinite-square potential. Each state has a volume of  $(\pi/L)^3$  in  $k$ -space.

eigenenergies of the wavefunctions are

$$-\frac{\hbar^2 \nabla^2}{2m} \psi = \frac{\hbar^2}{2m} \sum_i k_i^2 = \frac{\hbar^2 \pi^2}{2m L^2} (n_x^2 + n_y^2 + n_z^2) \quad (6)$$



and as a result of these restrictions, states in  $k$ -space are confined to the first quadrant (c.f. Fig. 3). Each state has a volume  $(\pi/L)^3$  of  $k$ -space. Thus as  $L \rightarrow \infty$ , the number of states with energies  $E(\mathbf{k}) < E < E(\mathbf{k}) + dE$  is

$$dZ' = \frac{(4\pi k^2 dk)/8}{(\pi/L)^3}. \quad (7)$$

Then, since  $E = \frac{\hbar^2 k^2}{2m}$ , so  $k^2 dk = \frac{m}{\hbar^2} \sqrt{2mE/\hbar^2} dE$

$$dZ = dZ'/L^3 = \frac{1}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2} dE. \quad (8)$$

or, the density of state per unit volume is

$$D(E) = \frac{dZ}{dE} = \frac{1}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}. \quad (9)$$

Up until now, we have ignored the properties of electrons. However, for the DOS, it is useful to recall that the electrons are spin-1/2 thus  $2S + 1 = 2$  electrons can fill each orbital or  $k$ -state, one of spin up the other spin down. If we account for this spin degeneracy in  $D$ , then

$$D(E) = \frac{1}{2\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}. \quad (10)$$

## 2.2 The Fermi Gas

### 2.2.1 $T = 0$ , The Pauli Principle

Electrons, as are all half-integer spin particles, are Fermions. Thus, by the Pauli Principle, no two of them may occupy the same state. For example, if we calculate the density of electrons per unit volume

$$n = \int_0^\infty D(E)f(E, T)dE, \quad (11)$$

where  $f(E, T)$  is the probability that a state of energy  $E$  is occupied, the factor  $f(E, T)$  must enforce this restriction. However,  $f$  is just the statistical factor; c.f. for classical particles

$$f(E, T) = e^{-E/k_B T} \text{ for classical particles,} \quad (12)$$

which for  $T = 0$  would require all the electrons to go into the ground state  $f(0, 0) = 1$ . Clearly, this violates the Pauli principle.

At  $T = 0$  we need to put just one particle in each state, starting from the lowest energy state, until we are out of particles. Since  $E \propto k^2$  in our simple square-well model, will fill up all

k-states until we reach some Fermi radius  $k_F$ , corresponding to some Fermi Energy  $E_F$

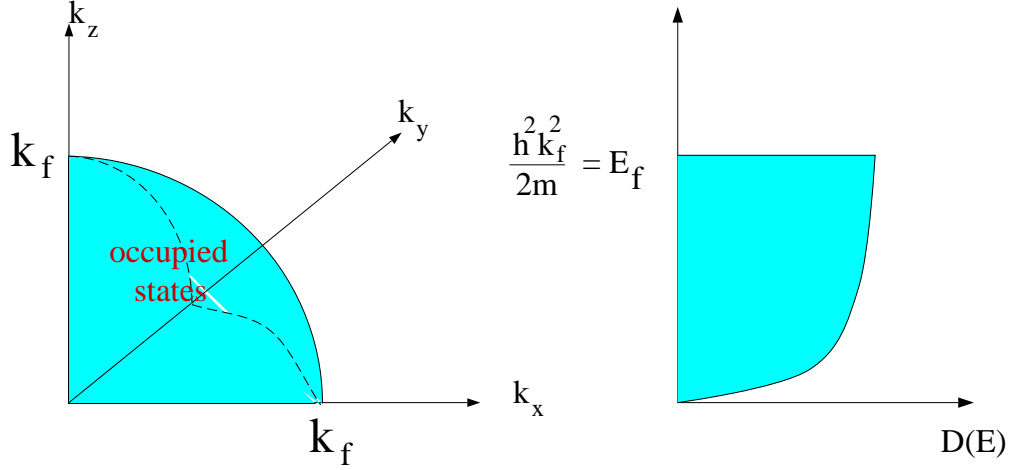


Figure 4: Due to the Pauli principle, all  $k$ -states up to  $k_F$ , and all states with energies up to  $E_f$  are filled at zero temperature.

$$E_F = \frac{\hbar^2 k_F^2}{2m}, \quad (13)$$

thus,

$$f(E, T = 0) = \theta(E_F - E) \quad (14)$$

and

$$\begin{aligned} n &= \int_0^\infty D(E) f(E, T) dE = \int_0^{E_F} D(E) DE \\ &= \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{1}{2\pi^2} \int_0^{E_F} E^{1/2} DE \\ &= \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{1}{2\pi^2} \frac{2}{3} E_F^{3/2}, \end{aligned} \quad (15)$$

or

$$E_F = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3} = k_B T_F \quad (16)$$

which also defines the Fermi temperature  $T_F$ . Thus for metals, in which  $n \approx 10^{23}/\text{cm}^3$ ,  $E_F \approx 10^{-11}\text{erg} \approx 10\text{eV} \approx k_B 10^5 K$ .

Notice that due to the Pauli principle, the average energy of the electrons will be finite, even at  $T = 0$ !

$$\mathcal{E} = \int_0^{E_F} D(E) E dE = \frac{3}{5} n E_F. \quad (17)$$

However, it is the electrons near  $E_F$  in energy which may be excited and are therefore important. These have a DeBroglie wavelength of roughly

$$\lambda_e = \frac{12.3 \text{ \AA}}{(E(\text{eV}))^{1/2}} \approx 4 \text{ \AA} \quad (18)$$

thus our original approximation of a square well potential, ignoring the lattice structure, is questionable for electrons near the Fermi surface, and should be regarded as yielding only qualitative results.

### 2.2.2 $T \neq 0$ , Fermi Statistics

At finite temperatures some of the states will be thermally excited. The energy available for these excitations is roughly  $k_B T$ , and the only possible excitations are from filled to unfilled electronic states. Therefore, only the states within  $k_B T$  ( $E_F - k_B T < E < E_F + k_B T$ ) of the Fermi surface may be excited.  $f(E, T)$  must be modified accordingly.

What we need is then  $f(E, T)$  at finite  $T$  which also satisfies the Pauli principle. Let's return to our model of a periodic solid which is constructed by bringing individual atoms together from an infinite separation. First, just consider a solid constructed from only two atoms, each with a single orbital (Fig. 5). For

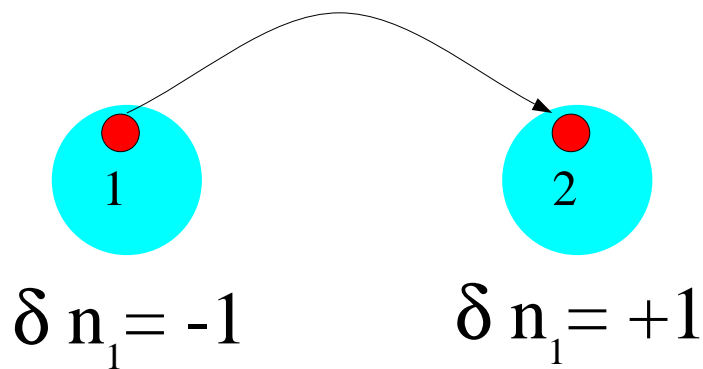


Figure 5: *Exchange of electrons in a solid composed of two orbitals.*

this system, in equilibrium,

$$0 = \delta F = \sum_i \frac{\partial F}{\partial n_i} \delta n_i \quad (19)$$

electrons are conserved so  $\sum_i \delta n_i = 0$ . Thus, for our two orbital system

$$\frac{\partial F}{\partial n_1} \delta n_1 + \frac{\partial F}{\partial n_2} \delta n_2 = 0 \quad \text{and} \quad \delta n_1 + \delta n_2 = 0 \quad (20)$$

or

$$\frac{\partial F}{\partial n_1} = \frac{\partial F}{\partial n_2} \quad (21)$$

A similar relation holds for an arbitrary number of particles. Apparently this quantity, the increased free energy needed to add a particle to the system, is a constant

$$\frac{\partial F}{\partial n_i} = \mu \quad (22)$$

for all  $i$ .  $\mu$  is called the *chemical potential*.

Now consider an ensemble of orbitals. We will treat the thermodynamics of this system within the canonical ensemble (i.e. the system is in contact with a thermal bath, and the particle number is conserved) for which  $\mathcal{F} = \mathcal{E} - T\mathcal{S}$  is the appropriate potential. The system energy  $\mathcal{E}$  and Entropy  $\mathcal{S}$  may be written

as functions of the orbital energies  $E_i$  and occupancies  $n_i$  and the degeneracy  $g_i$  of the state of energy  $E_i$ . For example,

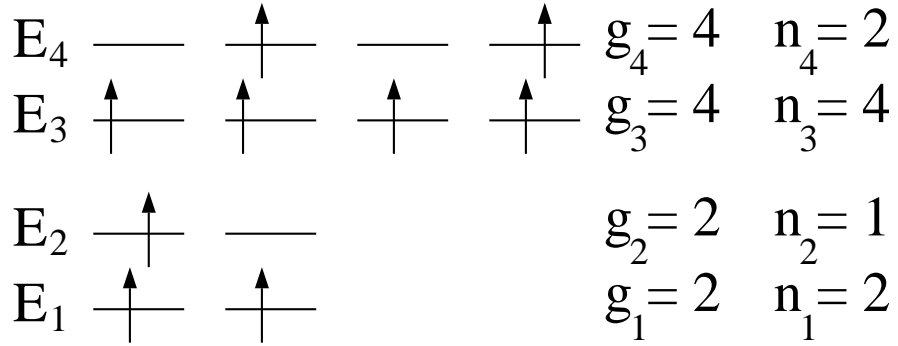


Figure 6: *states from an ensemble of orbitals.*

$$\mathcal{E} = \sum_i n_i E_i . \tag{23}$$

The entropy  $\mathcal{S}$  requires a bit more thought. If  $P$  is the number of ways of distributing the electrons among the states, then

$$\mathcal{S} = k_B \ln P . \tag{24}$$

Consider a set of  $g_i$  states with energy  $E_i$ . The number of ways of distributing the first electron in these states is  $g_i$ . For a second electron we then have  $g_i - 1$  ways... etc. So for  $n_i$  electrons there are

$$\frac{g_i!}{n_i!(g_i - n_i)!} \tag{25}$$

possible ways of accommodating the  $n_i$  (indistinguishable) electrons in  $g_i$  states.

The number of ways of making the whole system (ie, filling energy levels with  $E_i \neq E_j$ ) is then

$$P = \prod_i \frac{g_i!}{n_i!(g_i - n_i)!}, \quad (26)$$

and so, the entropy

$$\mathcal{S} = k_B \sum_i \ln g_i! - \ln n_i! - \ln(g_i - n_i)!. \quad (27)$$

For large  $n$ ,  $\ln n! \approx n \ln n - n$ , so

$$\mathcal{S} = k_B \sum_i g_i \ln g_i - n_i \ln n_i - (g_i - n_i) \ln(g_i - n_i) \quad (28)$$

and

$$\mathcal{F} = \sum_i n_i E_i - k_B T \sum_i g_i \ln g_i - n_i \ln n_i - (g_i - n_i) \ln(g_i - n_i) \quad (29)$$

We will want to use the chemical potential  $\mu$  in our thermodynamic calculations

$$\mu = \frac{\partial \mathcal{F}}{\partial n_k} = E_k + k_B T (\ln n_k + 1 - \ln(g_k - n_k) - 1), \quad (30)$$

where  $\beta = 1/k_B T$ . Solving for  $n_k$

$$n_k = \frac{g_k}{1 + e^{\beta(E_k - \mu)}}. \quad (31)$$



Thus the probability that a quantum state with energy  $E$  is occupied, is (the Fermi function)

$$f(E, T) = \frac{1}{1 + e^{\beta(E_k - \mu)}}. \quad (32)$$

At  $T = 0$ ,  $\beta = \infty$ , and  $f(E, 0) = \theta(\mu - E)$ . Thus  $\mu(T = 0) = E_F$ . However in general  $\mu$  is temperature dependent, since it must be adjusted to keep the particle number fixed. In addition,

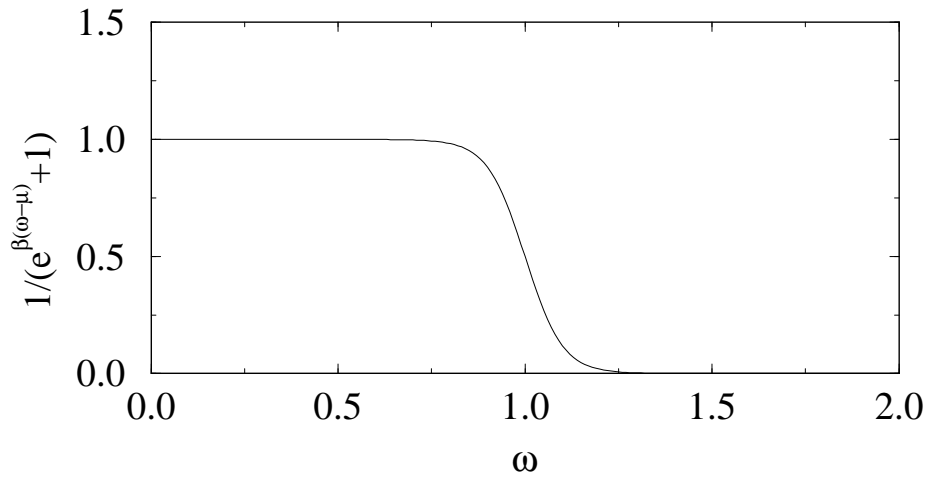


Figure 7: Plot of the Fermi function  $1/(e^{-\beta(\omega-\mu)} + 1)$  when  $\beta = 1/k_B T = 20$  and  $\mu = 1$ . Note that at energies  $\omega \approx \mu$  the Fermi function displays a smooth step of width  $\approx k_B T = 0.05$ . This allows thermal excitations of particles near the Fermi surface.

when  $T \neq 0$ ,  $f$  becomes less sharp at energies  $E \approx \mu$ . This reflects the fact that particles with energies  $E - \mu \approx k_B T$  may be excited to higher energy states.

**Specific Heat** The form of  $f(E, T)$  also clarifies why the electronic specific heat of metals is so small compared to the classical result  $C_{classical} = \frac{3}{2}nk_B T$ . The reason is simple: only the electrons with energies within about  $k_B T$  of the Fermi surface may be excited (about  $\frac{k_B T}{E_F}$  of the electron density) each with excitation energy of about  $k_B T$ . Therefore,

$$U_{excitation} \approx k_B T n \frac{k_B T}{E_F} = nk_B T \frac{T}{T_F} \quad (33)$$

so

$$C \approx nk_B \frac{T}{T_F} \quad (34)$$

Then as  $T \ll T_F$  ( $T_F$  is typically about  $10^5$ K in most metals<sup>2</sup>)  $C \approx nk_B \frac{T}{T_F} \ll C_{classical} \approx nk_B$ . Thus at temperatures where the phonons contribute essentially a classical result to the specific heat, the electronic contribution is vanishingly small. In general this holds except at very low  $T$  where the phonon contribution  $C_{phonon} \sim T^3$  goes to zero faster than the electronic contribution to the specific heat.

---

<sup>2</sup>Heavy Fermion systems are the exception to this rule. There  $T_F$  can be as small as a fraction of a degree Kelvin. As a result, they may have very large electronic specific heats.

**Specific Heat Calculation** Of course, since we know the free energy of the non-interacting Fermi gas, we can calculate the form of the specific heat. Here we will follow Ibach and Lüth and Kittel; however, since the chemical potential does depend upon the temperature, I would like to make the approximations we make a bit more explicit.

Upon heating from  $T = 0$  to finite  $T$ , the Fermi gas will gain energy

$$U(T) = \int_0^\infty dE E D(E) f(E, T) - \int_0^{E_F} dE E D(E) \quad (35)$$

so

$$C_V = \frac{dU}{dT} = \int_0^\infty dE E D(E) \frac{df(E, T)}{dT}. \quad (36)$$

Then since at constant volume the electronic density is constant, so  $\frac{dn}{dT} = 0$ , and  $n = \int_0^\infty dE D(E) f(E, T)$ ,

$$0 = E_F \frac{dn}{dT} = \int_0^\infty dE E_F D(E) \frac{df(E, T)}{dT} \quad (37)$$

so we may write

$$C_V = \int_0^\infty dE (E - E_F) D(E) \frac{df}{dT}. \quad (38)$$

In  $f$ , the temperature  $T$  enters through both  $\beta = 1/k_B T$  and

$\mu$

$$\begin{aligned} \frac{df}{dT} &= \frac{\partial f}{\partial \beta} \frac{\partial \beta}{\partial T} + \frac{\partial f}{\partial \mu} \frac{\partial \mu}{\partial T} \\ &= \frac{\beta e^{\beta(E-\mu)}}{(e^{\beta(E-\mu)} + 1)^2} \left[ \frac{E - \mu}{T} - \frac{\partial \mu}{\partial T} \right] \end{aligned} \quad (39)$$

However,  $\frac{\partial \mu}{\partial T}$  depends upon the details of the density of states near the Fermi surface, which can differ greatly from material to material. Furthermore,  $\frac{\partial \mu}{\partial T} < 1$  especially in common metals at temperatures  $T \ll T_F$ , and the first term  $\frac{E-\mu}{T}$  is of order one (c.f. Fig. 7). Thus, for now we will neglect  $\frac{\partial \mu}{\partial T}$  relative to  $\frac{E-\mu}{T}$  (you will explore the validity of this approximation in your homework), and, consistent with this approximation, replace  $\mu$  by  $E_F$ , so

$$\frac{df}{dT} \approx \frac{\beta e^{\beta(E-E_F)}}{(e^{\beta(E-E_F)} + 1)^2} \frac{E - E_F}{T}, \quad (40)$$

and

$$\begin{aligned} C_V &\approx \frac{1}{k_B T^2} \int_0^\infty dE \frac{(E - E_F)^2 D(E) \beta e^{\beta(E-E_F)}}{(e^{\beta(E-E_F)} + 1)^2} \quad \text{let } x = \beta(E - E_F) \\ &\approx k_B T \int_{-\beta E_F}^\infty dx D \left( \frac{x}{\beta} + E_F \right) x^2 \frac{e^x}{(e^x + 1)^2} \end{aligned} \quad (41)$$

As shown in Fig. 8, the function  $x^2 \frac{e^x}{(e^x+1)^2}$  is only large in the region  $-10 < x < 10$ . In this region, and for temperatures

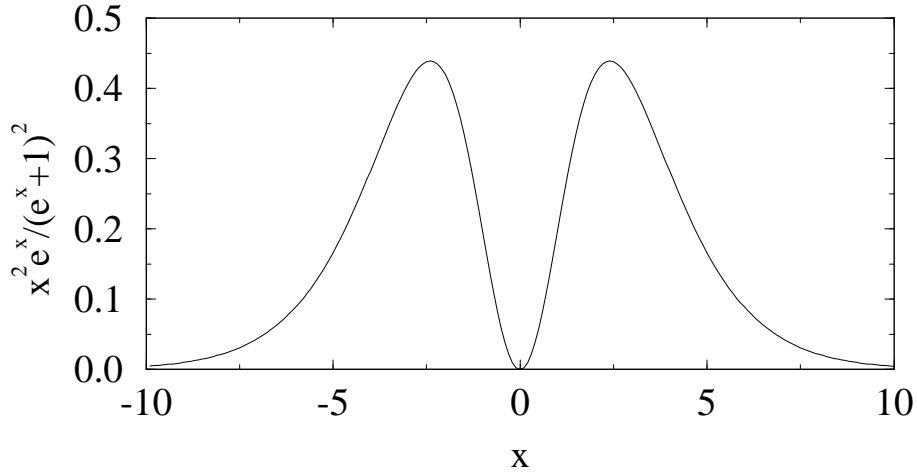


Figure 8: Plot of  $x^2 \frac{e^x}{(e^x+1)^2}$  vs.  $x$ . Note that this function is only finite for roughly  $-10 < x < 10$ . Thus, at temperatures  $T \ll T_F \sim 10^5 K$ , we can approximate  $D\left(\frac{x}{\beta} + E_F\right) \approx D(E_F)$  in Eq. 42.

$T \ll T_F$ ,  $D\left(\frac{x}{\beta} + E_F\right) \approx D(E_F)$ , since the density of states usually does not have features which are sharp on the energy scale of  $10k_B T$ . Thus

$$\begin{aligned}
 C_V &\approx k_B T D(E_F) \int_{-\beta E_F}^{\infty} dx x^2 \frac{e^x}{(e^x + 1)^2} \\
 &\approx \frac{\pi^2}{3} k_B^2 T D(E_F). \tag{42}
 \end{aligned}$$

Note that no assumption about the form of  $D(E)$  was made other than the assumption that it is smooth within  $k_B T$  of the Fermi surface. Thus, experimental measurements of the specific heat at constant volume of the electrons, gives us information

about the density of electronic states at the Fermi surface.

Now let's reconsider the DOS for the 3-D box potential.

$$D(E) = \frac{1}{2\pi^2} \left( \frac{2m}{\hbar} \right)^{3/2} E^{1/2} = D(E_F) \left( \frac{E}{E_F} \right)^{1/2} \quad (43)$$

For which  $n = \int_0^{E_F} D(E) dE = D(E_F) \frac{2}{3} E_F$ , so

$$C_V = \frac{\pi^2}{2} n k_B \frac{T}{T_F} \ll \frac{3}{2} n k_B \quad (44)$$

where the last term on the right is the classical result. For room temperatures  $T \sim 300\text{K}$ , which is also of the same order of magnitude as the Debye temperatures  $\theta_D$ ,

$$C_{V \text{ phonon}} \sim \frac{3}{2} n k_B \gg C_{V \text{ electron}} \quad (45)$$

So, the only way to measure the electronic specific heat in most materials is to go to very low temperatures  $T \ll \theta_D$ , for which  $C_{V \text{ phonon}} \sim T^3$ . Here the total specific heat

$$C_V \approx \gamma T + \beta T^3 \quad (46)$$

We will see that gives us some measurement of the electronic effective mass for our Fermi liquid theory. I.e. it tell us something about electron- electron interactions.

### 3 The Weakly Correlated Electronic Liquid

#### 3.1 Thomas-Fermi Screening

As an introduction to the effect of electronic correlations, consider the effect of a charged oxygen defect in one of the copper-oxygen planes of a cuprate superconductor shown in Fig. 9. Assume that the oxygen defect captures two electrons from the metallic band, going from a  $2s^2 2p^4$  to a  $2s^2 2p^6$  configuration. The defect will then become a cation, and have a net charge

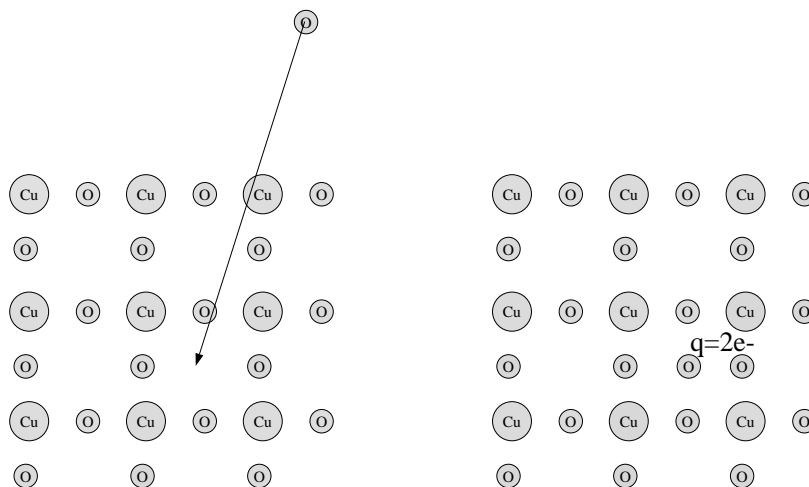


Figure 9: A charged oxygen defect is introduced into one of the copper-oxygen planes of a cuprate superconductor. The oxygen defect captures two electrons from the metallic band, going from a  $2s^2 2p^4$  to a  $2s^2 2p^6$  configuration.

of two electrons. In the vicinity of this oxygen defect, the elec-

trostatic potential and the electronic charge density will be reduced.

If we model the electronic density of states in this material with our box-potential DOS, we can think of this reduction in the local charge density in terms of raising the DOS parabola near the defect (cf. Fig. 10). This will cause the free electronic

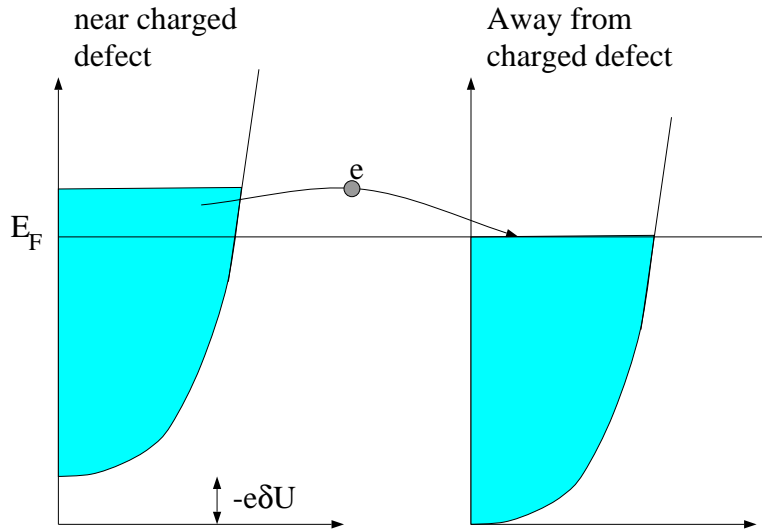


Figure 10: *The shift in the DOS parabola near a charged defect.*

charge to flow away from the defect. Near the defect (since  $e < 0$  and hence  $e\delta U(\mathbf{r}_{near}) < 0$ )

$$n(\mathbf{r}_{near}) \approx \int_0^{E_F + e\delta U(\mathbf{r}_{near})} D(E) DE \quad (47)$$



While away from the defect,  $\delta U(\mathbf{r}_{away}) = 0$ , so

$$n(\mathbf{r}_{away}) \approx \int_0^{E_F} D(E)DE \quad (48)$$

or

$$\delta n(\mathbf{r}) \approx \int_0^{E_F + e\delta U(\mathbf{r})} D(E)DE - \int_0^{E_F} D(E)DE \quad (49)$$

If  $|e\delta U| \ll E_F$ , then

$$\delta n(\mathbf{r}) \approx e\delta U D(E_F). \quad (50)$$

We can solve for the change in the electrostatic potential by solving Poisson equation.

$$\nabla^2 \delta U = 4\pi \delta \rho = 4\pi e \delta n = 4\pi e^2 D(E_F) \delta U. \quad (51)$$

Let  $\lambda^2 = 4\pi e^2 D(E_F)$ , then  $\nabla^2 \delta U = \lambda^2 \delta U$  has the solution<sup>3</sup>

$$\delta U(\mathbf{r}) = \frac{qe^{-\lambda r}}{r} \quad (52)$$

The length  $1/\lambda = r_{TF}$  is known as the Thomas-Fermi screening length.

$$r_{TF} = \left(4\pi e^2 D(E_F)\right)^{-1/2} \quad (53)$$

---

<sup>3</sup>The solution is actually  $Ce^{-\lambda r}/r$ , where  $C$  is a constant.  $C$  may be determined by letting  $D(E_F) = 0$ , so the medium in which the charge is embedded becomes vacuum. Then the potential of the charge is  $q/r$ , so  $C = q$ .

Lets estimate this distance for our square-well model,

$$r_{TF}^2 = \frac{a_0\pi}{3(3\pi^2n)^{1/3}} \approx \frac{a_0}{4n^{1/3}}$$

$$r_{TF} \approx \frac{1}{2} \left( \frac{n}{a_0^3} \right)^{-1/6} \quad (54)$$

In Cu, for which  $n \approx 10^{23} \text{ cm}^{-3}$  (and since  $a_0 = 0.53 \text{ \AA}$ )

$$r_{TFCu} \approx \frac{1}{2} \frac{(10^{23})^{-1/6}}{(0.5 \times 10^{-8})^{-1/2}} \approx 0.5 \times 10^{-8} \text{ cm} = 0.5 \text{ \AA} \quad (55)$$

Thus, if we add a charge defect to Cu metal, the effect of the defect's ionic potential is screened away for distances  $r > \frac{1}{2} \text{ \AA}$ .

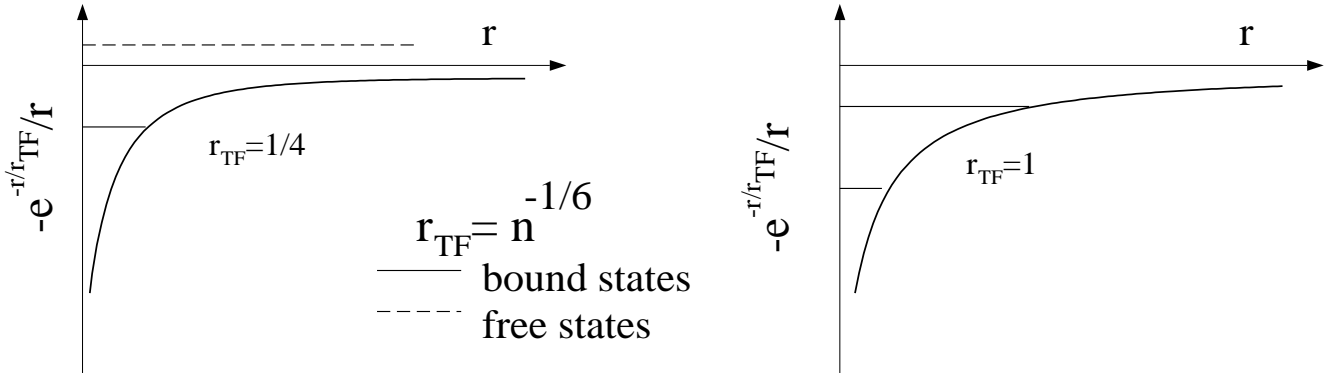


Figure 11: *Screened defect potentials. As the screening length increases, states that were free, become bound.*

Now consider an electron bound to an ion in Cu or some other metal. As shown in Fig. 11 the screening length decreases, and bound states rise up in energy. In a weak metal (i.e. something

like YBCO), in which the valence state is barely free, a reduction in the number of carriers (electrons) will increase the screening length, since

$$r_{TF} \sim n^{-1/6}. \quad (56)$$

This will extend the range of the potential, causing it to trap or bind more states—making the one free valence state bound.

Now imagine that instead of a single defect, we have a concentrated system of such ions, and suppose that we decrease the density of carriers (i.e. in Si-based semiconductors, this is done by doping certain compensating dopants, or even by modulating the pressure). This will in turn, increase the screening length, causing some states that were free to become bound, causing an abrupt transition from a metal to an insulator, and is believed to explain the MI transition in some transition-metal oxides, glasses, amorphous semiconductors, etc.

### 3.2 Fermi liquids

The purpose of these next several lectures is to introduce you to the theory of the Fermi liquid, which is, in its simplest form, a collection of Fermions in a box plus interactions.

In reality , the only physical analog is a gas of  $^3\text{He}$ , which due its nuclear spin (the nucleus has two protons, one neutron), obeys Fermi statistics for sufficiently low energies or temperatures. In addition, simple metals, from the first or second column of the periodic table, for which we may approximate the ionic potential

$$V(\mathbf{R}) = V_0 \tag{57}$$

are a close approximant to Fermi liquids.

Moreover, Fermi Liquid theory only describes the "gaseous" phase of these quantum fermion systems. For example,  $^3\text{He}$  also has a superfluid (triplet), and at least in  $^4\text{He}$ - $^3\text{He}$  mixtures, a solid phase exists which is not described by Fermi Liquid Theory. One should note; however, that the Fermi liquid theory state does serve as the starting point for the theories of super-

conductivity and super fluidity.

One may construct Fermi liquid theory either starting from a many-body diagrammatic or phenomenological viewpoint. We, as Landau, will choose the latter. Fermi liquid theory has 3 basic tenants:

1. momentum and spin remain good quantum numbers to describe the (quasi) particles.
2. the interacting system may be obtained by adiabatically turning on a particle-particle interaction over some time  $t$ .
3. the resulting excitations may be described as quasi-particles with lifetimes  $\gg t$ .

### **3.3 Quasi-particles**

The last assumption involves a new concept, that of the quasi-particles which requires some explanation.

#### **3.3.1 Particles and Holes**

Particles and Holes are excitations of the non-interacting system at zero temperature. Consider a system of  $N$  free Fermions

each of mass  $m$  in a volume  $V$ . The eigenstates are the anti-symmetrized combinations (Slater determinants) of  $N$  different single particle states.

$$\psi_{\mathbf{p}}(\mathbf{r}) = \frac{1}{\sqrt{V}} e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} \quad (58)$$

The occupation of each of these states is given by  $n_{\mathbf{p}} = \theta(p - p_F)$  where  $p_F$  is the radius of the Fermi sphere. The energy of the system is

$$E = \sum_{\mathbf{p}} n_{\mathbf{p}} \frac{p^2}{2m} \quad (59)$$

and  $p_F$  is given by

$$\frac{N}{V} = \frac{1}{3\pi^2} \left( \frac{p_F}{\hbar} \right)^3 \quad (60)$$

Now lets add a particle to the lowest available state  $p = p_F$  then, for  $T = 0$ ,

$$\mu = E_0(N + 1) - E_0(N) = \frac{\partial E_0}{\partial N} = \frac{p_F^2}{2m}. \quad (61)$$

If we now excite the system, we will promote a certain number of particles across the Fermi surface  $S_F$  yielding particles above and an equal number of vacancies or holes below the Fermi surface. These are our elementary excitations, and they are

quantified by  $\delta n_p = n_p - n_p^0$

$$\delta n_p = \begin{cases} \delta_{p,p'} & \text{for a particle } p' > p_F \\ -\delta_{p,p'} & \text{for a hole } p' < p_F \end{cases} . \quad (62)$$

If we consider excitations created by thermal fluctuations, then

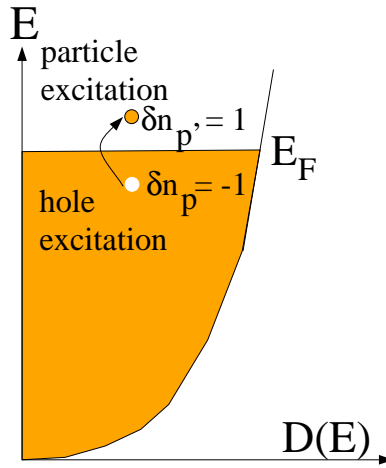


Figure 12: Particle and hole excitations of the Fermi gas.

$\delta n_p \sim 1$  only for excitations of energy within  $k_B T$  of  $E_F$ . The energy of the non-interacting system is completely characterized as a functional of the occupation

$$E - E_0 = \sum_{\mathbf{p}} \frac{p^2}{2m} (n_p - n_p^0) = \sum_{\mathbf{p}} \frac{p^2}{2m} \delta n_p . \quad (63)$$

Now lets take our system and place it in contact with a particle bath. Then the appropriate potential is the free energy,

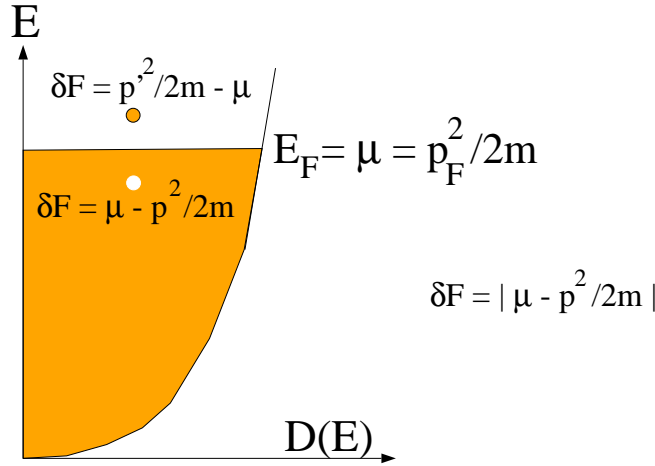


Figure 13: *Since  $\mu = p_F^2/2m$ , the free energy of a particle or a hole is  $\delta F = |p^2/2m - \mu| > 0$ , so the system is stable to these excitations.*

which for  $T = 0$ , is  $F = E - \mu N$ , and

$$F - F_0 = \sum_p \left( \frac{p^2}{2m} - \mu \right) \delta n_p. \quad (64)$$

The free energy of a particle, with momentum  $\mathbf{p}$  and  $\delta n_{\mathbf{p}'} = \delta_{\mathbf{p},\mathbf{p}'}$  is  $\frac{p^2}{2m} - \mu$  and it corresponds to an excitation outside  $S_F$ . The free energy of a hole  $\delta n_{\mathbf{p}'} = -\delta_{\mathbf{p},\mathbf{p}'}$  is  $\mu - \frac{p^2}{2m}$ , which corresponds to an excitation within  $S_F$ . However, since  $\mu = p_F^2/2m$ , the free energy of either at  $p = p_F$  is zero, hence the free energy of an excitation is

$$\left| p^2/2m - \mu \right|, \quad (65)$$

which is always positive; ie., the system is stable to excitations.



### 3.3.2 Quasiparticles and Quasiholes at $T = 0$

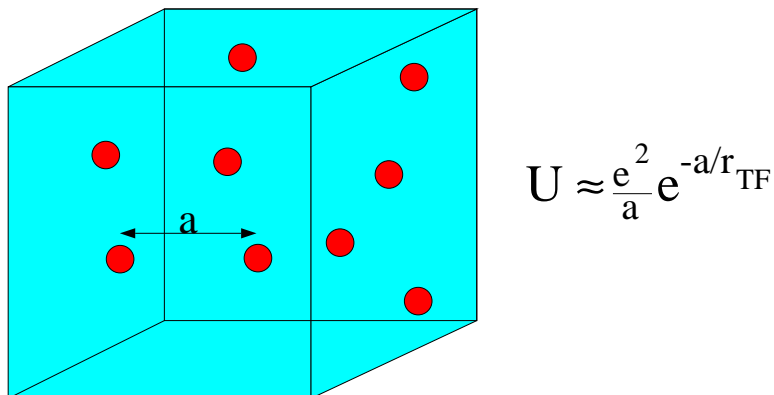


Figure 14: *Model for a fermi liquid: a set of interacting particles an average distance  $a$  apart bound within an infinite square-well potential.*

Now let's consider a system with interacting particles an average distance  $a$  apart, so that the characteristic energy of interaction is  $\frac{e^2}{a} e^{-a/r_{TF}}$ . We will imagine that this system evolves slowly from an ideal or noninteracting system in time  $t$  (i.e. the interaction  $U \approx \frac{e^2}{a} e^{-a/r_{TF}}$  is turned on slowly, so that the noninteracting system evolves while remaining in the ground state into an interacting system in time  $t$ ).

If the eigenstate of the ideal system is characterized by  $n_p^0$ , then the interacting system eigenstate will evolve quasistatistically from  $n_p^0$  to  $n_p$ . In fact if the system is isotropic and

remains in its ground state, then  $n_p^0 = n_p$ . However, clearly in some situations (superconductivity, magnetism) we will neglect some eigenstates of the interacting system in this way.

Now let's add a particle of momentum  $\mathbf{p}$  to the non-interacting ideal system, and slowly turn on the interaction. As  $U$  is

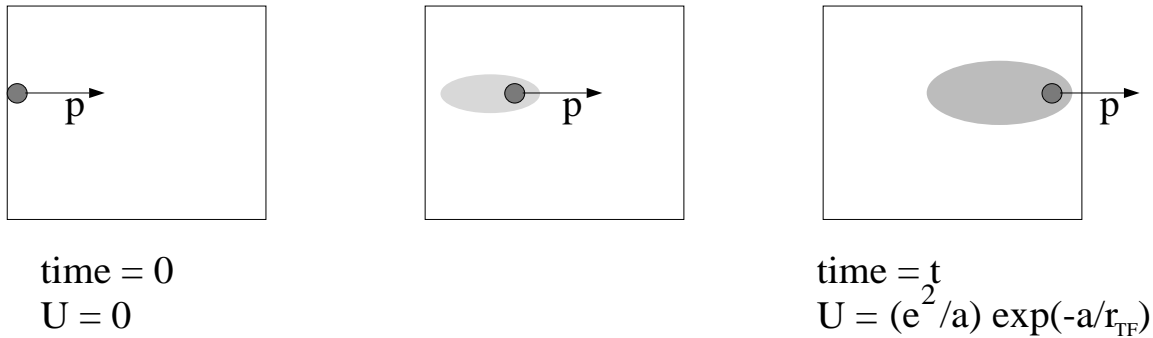


Figure 15: We add a particle with momentum  $\mathbf{p}$  to our noninteracting ( $U = 0$ ) Fermi liquid at time  $t = 0$ , and slowly increase the interaction to its full value  $U$  at time  $t$ . As the particle and system evolve, the particle becomes dressed by interactions with the system (shown as a shaded ellipse) which changes the effective mass but not the momentum of this single-particle excitation (now called a quasi-particle).

switched on, we slowly begin to perturb the particles close to the additional particle, so the particle becomes dressed by these interactions. However since momentum is conserved, we have created an excitation (particle and its cloud) of momentum  $\mathbf{p}$ . We call this particle and cloud a quasiparticle. In the same way,

if we had introduced a hole of momentum  $\mathbf{p}$  below the Fermi surface, and slowly turned on the interaction, we would have produced a quasihole.

Note that this adiabatic switching on procedure will have difficulties if the lifetime of the quasi-particle  $\tau < t$ . If so, then the the quasiparticle could decay into a number of other quasiparticles and quasiholes. If we shorten  $t$  so that again  $\tau \gg t$ , then switching on  $U$  could excite the system out of its ground state). We will show that such difficulties do not arise so long as the energy of the particle is close to the Fermi energy. Here there are few states accessible for creating particle-hole excitations so collisions are rare.

To estimate this lifetime consider the following argument from AGD: A particle with momentum  $\mathbf{p}_1$  above the Fermi surface ( $p_1 > p_F$ ) interacts with one of the particles below the Fermi surface with momentum  $\mathbf{p}_2$ . As a result, two new particles appear above the Fermi surface (all other states are full) with momenta  $\mathbf{p}_3$  and  $\mathbf{p}_4$ . This may also be interpreted as a particle of momentum  $\mathbf{p}_1$  decaying into particles with momenta

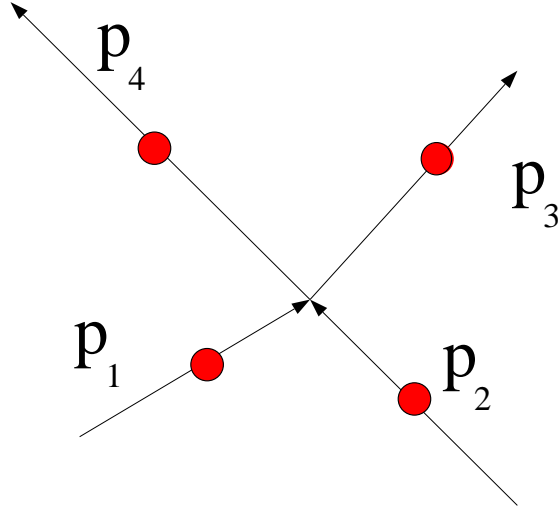


Figure 16: A particle with momentum  $\mathbf{p}_1$  above the Fermi surface ( $p_1 > p_F$ ) interacts with one of the particles below the Fermi surface with momentum  $\mathbf{p}_2$ . As a result, two new particles appear above the Fermi surface (all other states are full) with momenta  $\mathbf{p}_3$  and  $\mathbf{p}_4$ .

$\mathbf{p}_3$  and  $\mathbf{p}_4$  and a hole with momentum  $\mathbf{p}_2$ . By Fermi's golden rule, the total probability of such a process is proportional to

$$\frac{1}{\tau} \propto \int \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) d^3 p_2 d^3 p_3 \quad (66)$$

where  $\varepsilon_1 = \frac{p_1^2}{2m} - E_F$ , and the integral is subject to the constraints of energy and momentum conservation and that

$$p_2 < p_F, \quad p_3 > p_F, \quad p_4 = |\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3| > p_F \quad (67)$$

It must be that  $\varepsilon_1 + \varepsilon_2 = \varepsilon_3 + \varepsilon_4 > 0$  since both particles 3 and 4 must be above the Fermi surface. However, since  $\varepsilon_2 < 0$ ,

if  $\varepsilon_1$  is small, then  $|\varepsilon_2| \lesssim \varepsilon_1$  is also small, so only of order  $\varepsilon_1/E_F$  states may scatter with the state  $\mathbf{k}_1$ , conserve energy, and obey the Pauli principle. Thus, restricting  $\varepsilon_2$  to a narrow shell of width  $\varepsilon_1/E_F$  near the Fermi surface, and reducing the scattering probability  $1/\tau$  by the same factor.

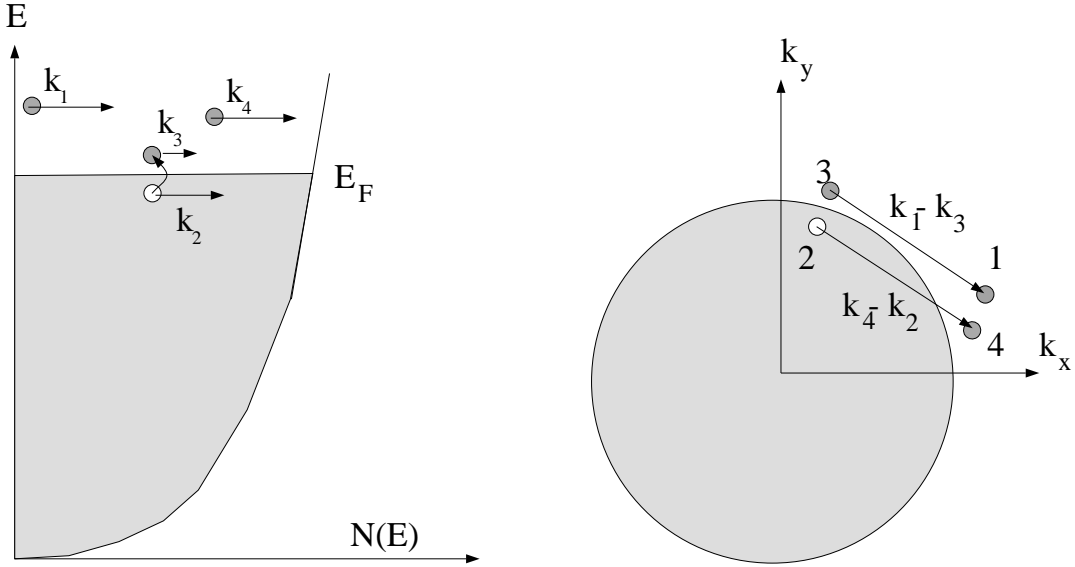


Figure 17: A quasiparticle of momentum  $\mathbf{p}_1$  decays via a particle-hole excitation into a quasiparticle of momentum  $\mathbf{p}_4$ . This may also be interpreted as a particle of momentum  $\mathbf{p}_1$  decaying into particles with momenta  $\mathbf{p}_3$  and  $\mathbf{p}_4$  and a hole with momentum  $\mathbf{p}_2$ . Energy conservation requires  $|\varepsilon_2| \lesssim \varepsilon_1$ . Thus, restricting  $\varepsilon_2$  to a narrow shell of width  $\varepsilon_1/E_F$  near the Fermi surface. Momentum conservation  $\mathbf{k}_1 - \mathbf{k}_3 = \mathbf{k}_4 - \mathbf{k}_2$  further restricts the available states by a factor of about  $\varepsilon_1/E_F$ . Thus the lifetime of a quasiparticle is proportional to  $\left(\frac{\varepsilon_1}{E_F}\right)^{-2}$ .

Now consider the constraints placed on states  $\mathbf{k}_3$  and  $\mathbf{k}_4$  by

momentum conservation

$$\mathbf{k}_1 - \mathbf{k}_3 = \mathbf{k}_4 - \mathbf{k}_2. \quad (68)$$

Since  $\varepsilon_1$  and  $\varepsilon_2$  are confined to a narrow shell around the Fermi surface, so too are  $\varepsilon_3$  and  $\varepsilon_4$ . This can be seen in Fig. 17, where the requirement that  $\mathbf{k}_1 - \mathbf{k}_3 = \mathbf{k}_4 - \mathbf{k}_2$  limits the allowed states for particles 3 and 4. If we take  $\mathbf{k}_1$  fixed, then the allowed states for 2 and 3 are obtained by rotating the vectors  $\mathbf{k}_1 - \mathbf{k}_3 = \mathbf{k}_4 - \mathbf{k}_2$ ; however, this rotation is severely limited by the fact that particle 3 must remain above, and particle 2 below, the Fermi surface. This restriction on the final states further reduces the scattering probability by a factor of  $\varepsilon_1/E_F$ .

Thus, the scattering rate  $1/\tau$  is proportional to  $\left(\frac{\varepsilon_1}{E_F}\right)^2$  so that excitations of sufficiently small energy will always be sufficiently long lived to satisfy the constraints of reversibility. Finally, the fact that the quasiparticle only interacts with a small number of other particles due to Thomas-Fermi screening (i.e. those within a distance  $\approx r_{TF}$ ), also significantly reduces the scattering rate.

### 3.4 Energy of Quasiparticles.

As in the non-interacting system, excitations will be quantified by the deviation of the occupation from the ground state occupation  $n_{\mathbf{p}}^0$

$$\delta n_{\mathbf{p}} = n_{\mathbf{p}} - n_{\mathbf{p}}^0. \quad (69)$$

At low temperatures  $\delta n_{\mathbf{p}} \sim 1$  only for  $p \approx p_F$  where the particles are sufficiently long lived that  $\tau \gg t$ . It is important to emphasize that only  $\delta n_{\mathbf{p}}$  not  $n_{\mathbf{p}}^0$  or  $n_{\mathbf{p}}$ , will be physically relevant. This is important since it does not make much sense to talk about quasiparticle states, described by  $n_{\mathbf{p}}$ , far from the Fermi surface since they are not stable.

For the ideal system

$$E - E_0 = \sum_{\mathbf{p}} \frac{p^2}{2m} \delta n_{\mathbf{p}}. \quad (70)$$

For the interacting system  $E[n_{\mathbf{p}}]$  becomes much more complicated. If however  $\delta n_{\mathbf{p}}$  is small (so that the system is close to its ground state) then we may expand:

$$E[n_{\mathbf{p}}] = E_o + \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} \delta n_{\mathbf{p}} + \mathcal{O}(\delta n_{\mathbf{p}}^2), \quad (71)$$

where  $\epsilon_{\mathbf{p}} = \delta E / \delta n_{\mathbf{p}}$ . Note that  $\epsilon_{\mathbf{p}}$  is intensive (ie. it is independent of the system volume). If  $\delta n_{\mathbf{p}} = \delta_{\mathbf{p},\mathbf{p}'}$ , then  $E \approx E_0 + \epsilon_{\mathbf{p}'}$ ; i.e., the energy of the quasiparticle of momentum  $\mathbf{p}'$  is  $\epsilon_{\mathbf{p}'}$ .

In practice we will only need  $\epsilon_p$  near the Fermi surface where  $\delta n_{\mathbf{p}}$  is finite. So we may approximate

$$\epsilon_{\mathbf{p}} \approx \mu + (\mathbf{p} - \mathbf{p}_F) \cdot \nabla_p \epsilon_{\mathbf{p}}|_{p_F} \quad (72)$$

where  $\nabla_p \epsilon_{\mathbf{p}} = v_{\mathbf{p}}$ , the group velocity of the quasiparticle. The ground state of the  $N + 1$  particle system is obtained by adding a particle with  $\epsilon_{\mathbf{p}} = \epsilon_F = \mu = \frac{\partial E_0}{\partial N}$  (at zero temperature); which defines the chemical potential  $\mu$ . We make learn more about  $\epsilon_{\mathbf{p}}$  by employing the symmetries of our system. If we explicitly display the spin-dependence,

$$\epsilon_{\mathbf{p},\sigma} = \epsilon_{-\mathbf{p},-\sigma} \text{ under time-reversal} \quad (73)$$

$$\epsilon_{\mathbf{p},\sigma} = \epsilon_{-\mathbf{p},\sigma} \text{ under BZ reflection} \quad (74)$$

So  $\epsilon_{\mathbf{p},\sigma} = \epsilon_{-\mathbf{p},\sigma} = \epsilon_{\mathbf{p},-\sigma}$ ; i.e. in the absence of an external magnetic field,  $\epsilon_{\mathbf{p},\sigma}$  does not depend upon  $\sigma$  if. Furthermore, for an isotropic system  $\epsilon_{\mathbf{p}}$  depends only upon the magnitude of  $\mathbf{p}$ ,  $|\mathbf{p}|$ , so  $\mathbf{p}$  and  $v_{\mathbf{p}} = \nabla \epsilon_{\mathbf{p}}(|\mathbf{p}|) = \frac{\mathbf{p}}{|\mathbf{p}|} \frac{d\epsilon_{\mathbf{p}}(|\mathbf{p}|)}{d|\mathbf{p}|}$  are parallel. Let us



define  $m^*$  as the constant of proportionality at the fermi surface

$$v_{p_F} = p_F/m^* \quad (75)$$

Using  $m^*$  it is useful to define the density of states at the fermi surface. Recall, that in the non-interacting system,

$$D(E_F) = \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} E_F^{1/2} = \frac{mp_F}{\pi\hbar^3} \quad (76)$$

where  $p = \hbar k$ , and  $E = p^2/2m$ . Thus, for the interacting system at the Fermi surface

$$D_{interacting}(E_F) = \frac{m^*p_F}{\pi\hbar^3}, \quad (77)$$

where the  $m^*$  (generally  $> m$ , but not always) accounts for the fact that the quasiparticle may be viewed as a dressed particle, and must “drag” this dressing along with it. I.e., the effective mass to some extent accounts for the interaction between the particles.

## 4 Interactions between Particles: Landau Fermi Liquid

### 4.1 The free energy, and interparticle interactions

The thermodynamics of the system depends upon the free energy  $F$ , which at zero temperature is

$$F - F_0 = E - E_0 - \mu(N - N_0). \quad (78)$$

Since our quasiparticles are formed by adiabatically switching on the interaction in the  $N + 1$  particle ideal system, adding one quasiparticle to the system adds one real particle. Thus,

$$N - N_0 = \sum_{\mathbf{p}} \delta n_{\mathbf{p}}, \quad (79)$$

and since

$$E - E_0 \approx \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} \delta n_{\mathbf{p}}, \quad (80)$$

we get

$$F - F_0 \approx \sum_{\mathbf{p}} (\epsilon_{\mathbf{p}} - \mu) \delta n_{\mathbf{p}}. \quad (81)$$

As shown in Fig. 18, we will be interested in excitations of the system which distort the Fermi surface by an amount proportional to  $\delta$ . For our theory/expansion to remain valid, we must

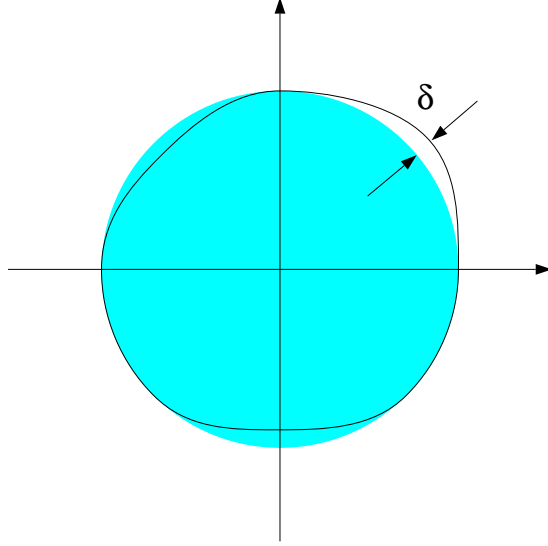


Figure 18: We consider small distortions of the fermi surface, proportional to  $\delta$ , so that  $\frac{1}{N} \sum_{\mathbf{p}} |\delta n_{\mathbf{p}}| \ll 1$ .

have

$$\frac{1}{N} \sum_{\mathbf{p}} |\delta n_{\mathbf{p}}| \ll 1. \quad (82)$$

Where  $\delta n_{\mathbf{p}} \neq 0$ ,  $\epsilon_{\mathbf{p}} - \mu$  will also be of order  $\delta$ . Thus,

$$\sum_{\mathbf{p}} (\epsilon_{\mathbf{p}} - \mu) \delta n_{\mathbf{p}} \sim \mathcal{O}(\delta^2), \quad (83)$$

so, to be consistent we must add the next term in the Taylor series expansion of the energy to the expression for the free energy.

$$F - F_0 = \sum_{\mathbf{p}} (\epsilon_{\mathbf{p}} - \mu) \delta n_{\mathbf{p}} + \frac{1}{2} \sum_{\mathbf{p}, \mathbf{p}'} f_{\mathbf{p}, \mathbf{p}'} \delta n_{\mathbf{p}} \delta n_{\mathbf{p}'} + \mathcal{O}(\delta^3) \quad (84)$$

where

$$f_{\mathbf{p},\mathbf{p}'} = \frac{\delta E}{\delta n_{\mathbf{p}}\delta n_{\mathbf{p}'}} \quad (85)$$

The term, proportional to  $f_{\mathbf{p},\mathbf{p}'}$ , was added (to the Sommerfeld theory) by L.D. Landau. Since each sum over  $\mathbf{p}$  is proportional to the volume  $V$ , as is  $F$ , it must be that  $f_{\mathbf{p},\mathbf{p}'} \sim 1/V$ . However, it is also clear that  $f_{\mathbf{p},\mathbf{p}'}$  is an interaction between quasiparticles, each of which is spread out over the whole volume  $V$ , so the probability that they will interact is  $\sim r_{TF}^3/V$ , thus

$$f_{\mathbf{p},\mathbf{p}'} \sim r_{TF}^3/V^2 \quad (86)$$

In general, since  $\delta n_{\mathbf{p}}$  is only of order one near the Fermi surface, we will only care about  $f_{\mathbf{p},\mathbf{p}'}$  on the Fermi surface (assuming that it is continuous and changes slowly as we cross the Fermi surface).

$$\text{Interested in } f_{\mathbf{p},\mathbf{p}'}|_{\epsilon_{\mathbf{p}}=\epsilon_{\mathbf{p}'}=\mu} \text{ in only!} \quad (87)$$

Thus,  $f_{\mathbf{p},\mathbf{p}'}$  only depends upon the angle between  $\mathbf{p}$  and  $\mathbf{p}'$ .

We can also reduce the spin dependence of  $f_{\mathbf{p},\mathbf{p}'}$  to a symmetric and anti symmetric part. First in the absence of an external

field, the system should be invariant under time-reversal, so

$$f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} = f_{-\mathbf{p}-\sigma,-\mathbf{p}'-\sigma'}, \quad (88)$$

and, in a system with reflection symmetry

$$f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} = f_{-\mathbf{p}\sigma,-\mathbf{p}'\sigma'}. \quad (89)$$

Then

$$f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} = f_{\mathbf{p}-\sigma,\mathbf{p}'-\sigma'}. \quad (90)$$

It must be then that  $f$  depends only upon the relative orientations of the spins  $\sigma$  and  $\sigma'$ , so there are only two independent components  $f_{\mathbf{p}\uparrow,\mathbf{p}'\uparrow}$  and  $f_{\mathbf{p}\uparrow,\mathbf{p}'\downarrow}$ . We can split these into symmetric and antisymmetric parts.

$$f_{\mathbf{p},\mathbf{p}'}^a = \frac{1}{2} (f_{\mathbf{p}\uparrow,\mathbf{p}'\uparrow} - f_{\mathbf{p}\uparrow,\mathbf{p}'\downarrow}) \quad f_{\mathbf{p},\mathbf{p}'}^s = \frac{1}{2} (f_{\mathbf{p}\uparrow,\mathbf{p}'\uparrow} + f_{\mathbf{p}\uparrow,\mathbf{p}'\downarrow}). \quad (91)$$

$f_{\mathbf{p},\mathbf{p}'}^a$  may be interpreted as an exchange interaction, or

$$f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} = f_{\mathbf{p},\mathbf{p}'}^s + \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' f_{\mathbf{p},\mathbf{p}'}^a \quad (92)$$

where  $\boldsymbol{\sigma}$  and  $\boldsymbol{\sigma}'$  are the Pauli matrices for the spins.

Our ideal system is isotropic in momentum. Thus,  $f_{\mathbf{p},\mathbf{p}'}^a$  and  $f_{\mathbf{p},\mathbf{p}'}^s$  will only depend upon the angle  $\theta$  between  $\mathbf{p}$  and  $\mathbf{p}'$ , and

so we may expand either  $f_{\mathbf{p},\mathbf{p}'}^a$  and  $f_{\mathbf{p},\mathbf{p}'}^s$

$$f_{\mathbf{p},\mathbf{p}'}^\alpha = \sum_{l=0}^{\infty} f_l^\alpha P_l(\cos \theta). \quad (93)$$

Conventionally these  $f$  parameters are expressed in terms of reduced units.

$$D(E_F) f_l^\alpha = \frac{V m^* p_F}{\pi^2 \hbar^3} f_l^\alpha = F_l^\alpha. \quad (94)$$

## 4.2 Local Energy of a Quasiparticle

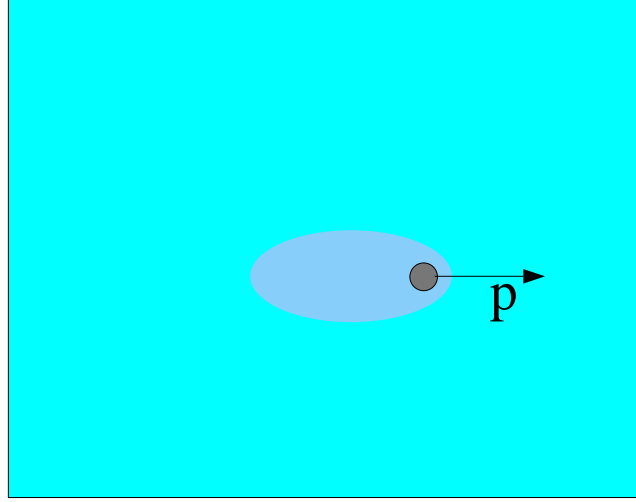


Figure 19: *The addition of another particle to a homogeneous system will yeilds in forces on the quasiparticle which tend to restore equilibrium.*

Now consider an interacting system with a certain distribution of excited quasiparticles  $\delta n_{\mathbf{p}'}$ . To this, add another quasiparticle of momentum  $\mathbf{p}$  ( $\delta n'_{\mathbf{p}} \rightarrow \delta n'_{\mathbf{p}} + \delta_{\mathbf{p},\mathbf{p}'}$ ). From Eq. 84 the

free energy of the additional quasiparticle is

$$\tilde{\epsilon}_{\mathbf{p}} - \mu = \epsilon_{\mathbf{p}} - \mu + \sum_{\mathbf{p}'} f_{\mathbf{p}',\mathbf{p}} \delta n_{\mathbf{p}'}, \quad (95)$$

(recall that  $f_{\mathbf{p},\mathbf{p}'} = f_{\mathbf{p}',\mathbf{p}}$ ). Both terms here are  $\mathcal{O}(\delta)$ . The second term describes the free energy of a quasiparticle due to the other quasiparticles in the system (some sort of Hartree-like term).

The term  $\tilde{\epsilon}_{\mathbf{p}}$  plays the part of the *local energy of a quasiparticle*. For example, the gradient of  $\tilde{\epsilon}_{\mathbf{p}}$  is the force the system exerts on the additional quasiparticle. When the quasiparticle is added to the system, the system is inhomogeneous so that  $\delta n_{\mathbf{p}'} = \delta n_{\mathbf{p}'}(\mathbf{r})$ . The system will react to this inhomogeneity by minimizing its free energy so that  $\nabla_r F = 0$ . However, only the additional free energy due the added particle (Eq. 95) is inhomogeneous, and has a non-zero gradient. Thus, the system will exert a force

$$-\nabla_r \tilde{\epsilon} = -\nabla_r \sum_{\mathbf{p}'} f_{\mathbf{p}',\mathbf{p}} \delta n_{\mathbf{p}'}(\mathbf{r}) \quad (96)$$

on the added quasiparticle resulting from interactions with other quasiparticles.

#### 4.2.1 Equilibrium Distribution of Quasiparticles at Finite $T$

$\tilde{\epsilon}_{\mathbf{p}}$  also plays an important role in the finite-temperature properties of the system. If we write

$$E - E_0 = \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} \delta n_{\mathbf{p}} + \frac{1}{2} \sum_{\mathbf{p}, \mathbf{p}'} f_{\mathbf{p}', \mathbf{p}} \delta n_{\mathbf{p}'} \delta n_{\mathbf{p}} \quad (97)$$

Now suppose that  $\sum_{\mathbf{p}} |\langle \delta n_{\mathbf{p}} \rangle| \ll N$ , as needed for the expansion above to be valid, so that

$$\delta n_{\mathbf{p}} = \langle \delta n_{\mathbf{p}} \rangle + (\delta n_{\mathbf{p}} - \langle \delta n_{\mathbf{p}} \rangle) \quad (98)$$

where the first term is  $\mathcal{O}(\delta)$ , and the second  $\mathcal{O}(\delta^2)$ . Thus,

$$\delta n_{\mathbf{p}} \delta n_{\mathbf{p}'} \approx -\langle \delta n_{\mathbf{p}} \rangle \langle \delta n_{\mathbf{p}'} \rangle + \langle \delta n_{\mathbf{p}} \rangle \delta n_{\mathbf{p}'} + \langle \delta n_{\mathbf{p}'} \rangle \delta n_{\mathbf{p}} \quad (99)$$

We may use this to rewrite the energy of our interacting system

$$\begin{aligned} E - E_0 &\approx \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} \delta n_{\mathbf{p}} - \frac{1}{2} \sum_{\mathbf{p}, \mathbf{p}'} f_{\mathbf{p}', \mathbf{p}} \langle \delta n_{\mathbf{p}} \rangle \langle \delta n_{\mathbf{p}'} \rangle + \sum_{\mathbf{p}, \mathbf{p}'} f_{\mathbf{p}', \mathbf{p}} \langle \delta n_{\mathbf{p}} \rangle \delta n_{\mathbf{p}'} \\ &\approx \sum_{\mathbf{p}} \left( \epsilon_{\mathbf{p}} + \sum_{\mathbf{p}'} f_{\mathbf{p}', \mathbf{p}} \langle \delta n_{\mathbf{p}'} \rangle \right) \delta n_{\mathbf{p}} - \frac{1}{2} \sum_{\mathbf{p}, \mathbf{p}'} f_{\mathbf{p}', \mathbf{p}} \langle \delta n_{\mathbf{p}} \rangle \langle \delta n_{\mathbf{p}'} \rangle \\ &\approx \sum_{\mathbf{p}} \langle \tilde{\epsilon}_{\mathbf{p}} \rangle \delta n_{\mathbf{p}} - \frac{1}{2} \sum_{\mathbf{p}, \mathbf{p}'} f_{\mathbf{p}', \mathbf{p}} \langle \delta n_{\mathbf{p}} \rangle \langle \delta n_{\mathbf{p}'} \rangle + \mathcal{O}(\delta^4) \end{aligned} \quad (100)$$

At this point, we may repeat the arguments made earlier to determine the fermion occupation probability for non-interacting



Fermions (the constant factor on the right hand-side has no effect). We will obtain

$$n_{\mathbf{p}}(T, \mu) = \frac{1}{1 + \exp \beta(\langle \tilde{\epsilon}_{\mathbf{p}} \rangle - \mu)}, \quad (101)$$

or

$$\delta n_{\mathbf{p}}(T, \mu) = \frac{1}{1 + \exp \beta(\langle \tilde{\epsilon}_{\mathbf{p}} \rangle - \mu)} - \theta(p_f - p). \quad (102)$$

However, at least for an isotropic system, this expression bears closer investigation. Here, the molecular field (evaluated within  $k_B T$  of the Fermi surface)

$$\langle \tilde{\epsilon}_{\mathbf{p}} - \epsilon_{\mathbf{p}} \rangle = \sum_{\mathbf{p}'} f_{\mathbf{p}', \mathbf{p}} \langle \delta n_{\mathbf{p}'} \rangle \quad (103)$$

must be independent of the location of  $\mathbf{p}$  on the Fermi surface (and of course, spin), and is thus constant. To see this, reconsider the Legendre polynomial expansion discussed earlier

$$\begin{aligned} \langle \tilde{\epsilon}_{\mathbf{p}} - \epsilon_{\mathbf{p}} \rangle &= \sum_{\mathbf{p}'} f_{\mathbf{p}', \mathbf{p}} \langle \delta n_{\mathbf{p}'} \rangle \\ &\propto \sum_l \int d^3 p f_l P_l(\cos \theta) \langle \delta n_{\mathbf{p}'} \rangle \\ &\propto f_0 \int d^3 p \langle \delta n_{\mathbf{p}'} \rangle = 0 \end{aligned} \quad (104)$$

In going from the second to the third line above, we made use of the isotropy of the system, so that  $\langle \delta n_{\mathbf{p}'} \rangle$  is independent of the angle  $\theta$ . The evaluation in the third line, follows from particle number conservation. Thus, to lowest order in  $\delta$

$$n_{\mathbf{p}}(T, \mu) = \frac{1}{1 + \exp \beta(\epsilon_{\mathbf{p}} - \mu)} + \mathcal{O}(\delta^4) \quad (105)$$

#### 4.2.2 Local Equilibrium Distribution

Now suppose we introduce a local weak perturbation such as the type discussed in Sec. 4.2 to an isotropic system at zero temperature. Such a perturbation could be caused by, eg. a sound wave or a weak magnetic field (which you will explore in your homework, to calculate the sound velocity and susceptibility of a Landau Fermi liquid). This perturbation will cause a small deviation of the equilibrium distribution function, leading to a new "local equilibrium" distribution

$$\bar{n}_{\mathbf{p}} = n_{\mathbf{p}}(\tilde{\epsilon}_{\mathbf{p}} - \mu) \quad (106)$$

where the argument of the RHS indicates that this is the distribution corresponding to the local energy discussed above. The

gradient of the local energy yields a force which tries to restore the equilibrium distribution  $n_{\mathbf{p}}(\epsilon_{\mathbf{p}} - \mu)$  derived above). The deviation from true equilibrium is

$$\delta n_{\mathbf{p}} = n_{\mathbf{p}} - \bar{n}_{\mathbf{p}} \quad (107)$$

$$= \delta \bar{n}_{\mathbf{p}} + \frac{\partial n_{\mathbf{p}}(\epsilon_{\mathbf{p}} - \mu)}{\partial \epsilon_{\mathbf{p}}} (\tilde{\epsilon}_{\mathbf{p}} - \epsilon_{\mathbf{p}}) \quad (108)$$

Using Eq. 95, we find

$$\delta n_{\mathbf{p}} = \delta \bar{n}_{\mathbf{p}} - \frac{\partial n_{\mathbf{p}}}{\partial \epsilon_{\mathbf{p}}} \sum_{\mathbf{p}'} f_{\mathbf{p},\mathbf{p}'} \delta n_{\mathbf{p}'} \quad (109)$$

At zero temperature, the factor  $\frac{\partial n_{\mathbf{p}}}{\partial \epsilon_{\mathbf{p}}} = -\delta(\epsilon_{\mathbf{p}} - \mu)$ , so both  $\delta n_{\mathbf{p}}$  and  $\delta \bar{n}_{\mathbf{p}}$  are restricted to the fermi surface. Since the perturbation of interest is small, we may expand both  $\delta n_{\mathbf{p}}$  and  $\delta \bar{n}_{\mathbf{p}}$  in a series of Legendre polynomials, and we will also split them into symmetric and antisymmetric parts (as we did with  $f_{\mathbf{p},\mathbf{p}'}$  previously). For example,

$$\delta n_{\mathbf{p}}^s = \sum_l \delta(\epsilon_{\mathbf{p}} - \mu) \delta n_l^s P_l \quad (110)$$

If we make a similar expansion for the antisymmetric and symmetric parts of  $\delta \bar{n}_{\mathbf{p}}$ , and substitute this back into Eq. 108, then

we find

$$\delta\bar{n}_l^a = \left(1 + \frac{F_l^a}{2l+1}\right) \delta n_l^a \quad (111)$$

$$\delta\bar{n}_l^s = \left(1 + \frac{F_l^s}{2l+1}\right) \delta n_l^s \quad (112)$$

### 4.3 Effective Mass $m^*$ of Quasiparticles

This argument most closely follows that of AGD, and we will follow their notation as closely as possible (without introducing any new symbols). In particular, since an integration by parts is necessary, we will use a momentum integral (as opposed to a momentum sum) notation

$$\sum_{\mathbf{p}} \rightarrow V \int \frac{d^3p}{(2\pi\hbar)^3}. \quad (113)$$

The net momentum of the volume  $V$  of quasiparticles is

$$\mathbf{P}_{qp} = 2V \int \frac{d^3p}{(2\pi\hbar)^3} \mathbf{p} n_{\mathbf{p}} \text{ net quasiparticle momentum} \quad (114)$$

which is also the momentum of the Fermi liquid. On the other hand since the number of particles equals the number of quasiparticles, the quasiparticle and particle currents must also be

equal

$$\mathbf{J}_{qp} = \mathbf{J}_p = 2V \int \frac{d^3p}{(2\pi\hbar)^3} \mathbf{v}_p n_p \text{ net quasiparticle and particle current} \quad (115)$$

or, since the momentum is just the particle mass times this current

$$\mathbf{P}_p = 2Vm \int \frac{d^3p}{(2\pi\hbar)^3} \mathbf{v}_p n_p \text{ net quasiparticle and particle current} \quad (116)$$

where  $\mathbf{v}_p = \nabla_p \tilde{\epsilon}_p$ , is the velocity of the quasiparticle. So

$$\int \frac{d^3p}{(2\pi\hbar)^3} \mathbf{p} n_p = m \int \frac{d^3p}{(2\pi\hbar)^3} \nabla_p \tilde{\epsilon}_p n_p \quad (117)$$

Now make an arbitrary change of  $n_p$  and recall that  $\tilde{\epsilon}_p$  depends upon  $n_p$ , so that

$$\delta \tilde{\epsilon}_p = V \sum_{\sigma'} \int \frac{d^3p'}{(2\pi\hbar)^3} f_{p,p'} \delta n_{p'}. \quad (118)$$

For Eq. 117, this means that

$$\begin{aligned} \int \frac{d^3p}{(2\pi\hbar)^3} \mathbf{p} \delta n_p &= m \int \frac{d^3p}{(2\pi\hbar)^3} \nabla_p \tilde{\epsilon}_p \delta n_p \\ &+ mV \int \frac{d^3p}{(2\pi\hbar)^3} \sum_{\sigma'} \int \frac{d^3p'}{(2\pi\hbar)^3} \nabla_p (f_{p,p'} \delta n_{p'}) n_p, \end{aligned} \quad (119)$$

or integrating by parts (and renaming  $\mathbf{p} \rightarrow \mathbf{p}'$  in the last part), we get

$$\begin{aligned} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{\mathbf{p}}{m} \delta n_{\mathbf{p}} &= \int \frac{d^3p}{(2\pi\hbar)^3} \nabla_p \tilde{\epsilon}_{\mathbf{p}} \delta n_{\mathbf{p}} \\ &\quad - V \sum_{\sigma'} \int \frac{d^3p'}{(2\pi\hbar)^3} \int \frac{d^3p}{(2\pi\hbar)^3} \delta n_{\mathbf{p}} f_{\mathbf{p},\mathbf{p}'} \nabla_{p'} n_{\mathbf{p}'} , \end{aligned} \quad (120)$$

Then, since  $\delta n_{\mathbf{p}}$  is arbitrary, it must be that the integrands themselves are equal

$$\frac{\mathbf{p}}{m} = \nabla_p \tilde{\epsilon}_{\mathbf{p}} - \sum_{\sigma'} V \int \frac{d^3p'}{(2\pi\hbar)^3} f_{\mathbf{p},\mathbf{p}'} \nabla_{p'} n_{\mathbf{p}'} \quad (121)$$

The factor  $\nabla_{p'} n_{\mathbf{p}'} = -\frac{\mathbf{p}'}{p'} \delta(p' - p_F)$ . The integral may be evaluated by taking advantage of the system isotropy, and setting  $\mathbf{p}$  parallel to the  $z$ -axis, since we are mostly interested in the properties of the system on the Fermi surface we take  $p = p_F$ , let  $\theta$  be the angle between  $\mathbf{p}$  (or the  $z$ -axis) and  $\mathbf{p}'$ , and finally note that on the Fermi surface  $|\nabla_p \tilde{\epsilon}_p|_{p=p_F} = v_F = p_F/m^*$ . Thus,

$$\frac{p_F}{m} = \frac{p_F}{m^*} + \sum_{\sigma'} \int \frac{p'^2 dp d\Omega}{(2\pi\hbar)^3} f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} \frac{\mathbf{p}'}{p'} \delta(p' - p_F) \quad (122)$$

However, since both  $p$  and  $p'$  are restricted to the Fermi surface  $\frac{p'}{p} = \cos \theta$ , and evaluating the integral over  $p$ , we get

$$\frac{1}{m} = \frac{1}{m^*} + \frac{V p_F}{2} \sum_{\sigma, \sigma'} \int \frac{d\Omega}{(2\pi\hbar)^3} f_{\mathbf{p}\sigma, \mathbf{p}'\sigma'} \cos \theta, \quad (123)$$

where the additional factor of  $\frac{1}{2}$  compensates for the additional spin sum. If we now sum over both spins,  $\sigma$  and  $\sigma'$ , only the symmetric part of  $f$  survives (the sum yields  $4f^s$ ), so

$$\frac{1}{m} = \frac{1}{m^*} + \frac{4\pi V p_F}{(2\pi\hbar)^3} \int d(\cos \theta) f^s(\theta) \cos \theta, \quad (124)$$

We now expand  $f$  in a Legendre polynomial series

$$f^\alpha(\theta) = \sum_l f_l^\alpha P_l(\cos \theta), \quad (125)$$

and recall that  $P_0(x) = 1$ ,  $P_1(x) = x$ , .... that

$$\int_{-1}^1 dx P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{nm} \quad (126)$$

and finally that

$$D(0) f_l^\alpha = \frac{V m^* p_F}{\pi^2 \hbar^3} f_l^\alpha = F_l^\alpha, \quad (127)$$

we find that

$$\frac{1}{m} = \frac{1}{m^*} + \frac{F_1^s}{3m^*}, \quad (128)$$

Quantity	Fermi Liquid	Fermi Liquid/Fermi Gas
Specific Heat	$C_v = \frac{m^* p_F}{3\hbar^3} k_B^2 T$	$\frac{C_V}{C_{V0}} = \frac{m^*}{m} = 1 + F_1^s/3$
Compressibility		$\frac{\kappa}{\kappa_0} = \frac{1+F_0^s}{1+F_0^s/3}$
Sound Velocity	$c^2 = \frac{p_F^2}{3mm^*} (1 + F_0^s)$	$\left(\frac{c}{c_0}\right)^2 = \frac{1+F_0^s}{1+F_1^s/3}$
Spin Susceptibility	$\chi = \frac{m^* p_F}{\pi^2 \hbar^3} \frac{\beta^2}{1+F_0^a}$	$\frac{\chi}{\chi_0} = \frac{1+F_1^s/3}{1+F_0^a}$

Table 1: *Fermi Liquid relations between the Landau parameters  $F_n^\alpha$  and some experimentally measurable quantities. For the latter, a zero subscript indicates the value for the non-interacting Fermi gas.*

or  $m^*/m = 1 + F_1^s/3$ .

The effective mass cannot be experimentally measured directly; however, it appears in many physically relevant measurable quantities, including the specific heat

$$C_V = \left( \frac{\partial E/V}{\partial T} \right)_{VN} = \frac{1}{V} \frac{\partial}{\partial T} \sum_{\mathbf{p}} \tilde{\epsilon}_{\mathbf{p}} n_{\mathbf{p}}. \quad (129)$$

To lowest order in  $\delta$ , we may neglect  $f_{\mathbf{p},\mathbf{p}'}$  in both  $\tilde{\epsilon}_{\mathbf{p}}$  and  $n_{\mathbf{p}}$ , so

$$C_V = \frac{1}{V} \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} \frac{\partial n_{\mathbf{p}}}{\partial T}. \quad (130)$$

Recall that the density of states  $D(E) = \sum_{\mathbf{p}} \delta(E - \epsilon_p)$ , and making the same assumption that we made for the non-interacting



system, that  $\frac{\partial\mu}{\partial T}$  is negligible, we get,

$$C_V = \frac{1}{V} \int d\epsilon D(\epsilon) \epsilon \frac{\partial}{\partial T} \frac{1}{\exp \beta(\epsilon - \mu) + 1}. \quad (131)$$

This integral is identical to the one we had to evaluate for the non-interacting system, and yields the result

$$\begin{aligned} C_V &= \frac{\pi^2}{3V} k_B^2 T D(E_F) \\ &= \frac{k_B^2 T m^* p_F}{3\hbar^3}. \end{aligned} \quad (132)$$

Thus, measuring the electronic contribution to the specific heat  $C_V$  yields information about the effective mass  $m^*$ , and hence  $F_1^s$ . Other measurements are related to some of the remaining Landau parameters, as summarized in table 1.